Matroid union (Korte & Vygen 13.6)

Let $M_i = (E, \mathcal{J}_i)$, $i = 1, 2, \dots, k$ be matroids over the same groundset E.

We call a set $X \subseteq E$ partitionable if $\exists\, X_1, X_2, \dots X_k$ disjoint sets s.t. $X_i \in \mathcal{J}_i$ and $X = X_1 \cup X_2 \cup \dots \cup X_k$

Let $\mathcal{J} = \{X \subseteq E \mid X \text{ is partitionable}\}$

Theorem 13.34 $M = (E, \mathcal{J})$ is a matroid with rank function $r(X) = \min\limits_{A \subseteq X} \{ |X - A| + \sum_{i=1}^{k} r_i(A) \}$

We also denote M be $M = \bigvee_{i=1}^{k} M_i$

Note By Theorem 13.34, if we have an oracle which can decide (in pol time) whether a given set is partitionable, then we can apply the greedy algorithm to find a maximum weight partitionable set X for a given $w : E \to \mathbb{R}_+$ and matroids $M_i = (E, \mathcal{J}_i)$ $i = 1, 2, \dots, k$.

Proof of Theorem 13.34:

We first show how to find, for a given subset $X \subseteq E$ a maximum size partitionable subset $X' \subseteq X$ and characterize its size.

Claim if we denote by $r(X)$ the maximum size of a subset $X^* \subseteq X$ which is partitionable, then we have

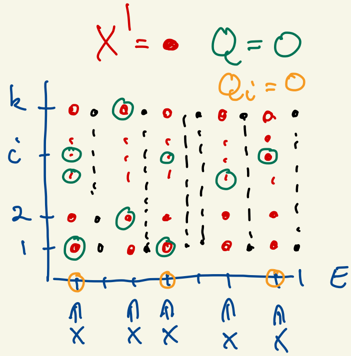
$$r(X) = \min_{A \subseteq X} \left\{ |X-A| + \sum_{i=1}^{k} r_i(A) \right\}$$

$\leq$: Let $Y \subseteq X$ be partitionable and let $Y = Y_1 \cup \ldots \cup Y_k$ be a partition with $Y_i \in \mathcal{J}_i$ for $i = 1, 2, \ldots, k$

Then for every $A \subseteq X$:

$$|Y| = |Y-A| + |Y \cap A|$$
$$\leq |X-A| + \sum_{i=1}^{k} |Y_i \cap A| \leq |X-A| + \sum_{i=1}^{k} r_i(A)$$

So $r(X) \leq \min_{A \subseteq X} \left\{ |X-A| + \sum_{i=1}^{k} r_i(A) \right\}$



$\geq$: Let $X' = \{(e,i) \mid e \in X \wedge i \in \{1, 2, \ldots, k\}\}$ and for each $Q \subseteq X'$ denote by Q_i $i \in [k]$ the set $Q_i = \{e \in X \mid (e,i) \in Q\}$

Let $\mathcal{J}_1 = \{Q \subseteq X' \mid Q_i \in \mathcal{J}_i \text{ for } i \in [k]\}$

$\mathcal{J}_2 = \{Q \subseteq X' \mid Q_i \cap Q_j = \emptyset \ \forall\, 1 \leq i < j \leq k\}$

$\mathcal{J}_1 = \{ Q \subseteq X' \mid Q_i \in \mathcal{J}_i \text{ for } i \in [k] \}$

$M' = (X', \mathcal{J}_1)$ is a matroid because each M_i is a matroid:

If $Q', Q \in \mathcal{J}_1$ with $|Q'| > |Q|$ then $|Q'_i| > |Q_i|$ for some i

so as $\mathcal{J}_i$ satisfies the exchange property, there is some $e \in Q'_i - Q_i$

such that $Q_i + e \in \mathcal{J}_i$.

This shows that $Q + (e,i) \in \mathcal{J}_1$ so $\mathcal{J}_1$ satisfies the exchange property.

The rank function s_1 of M' is $s_1(Q) = \sum_{i=1}^{k} r_i(Q_i)$:

if $\hat{Q} \subseteq Q$ is a maximum independent subset of Q then

each $\hat{Q}_i \subseteq Q_i$ is a maximum independent subset of Q_i in M_i so

$$|\hat{Q}| = \sum_{i=1}^{k} |\hat{Q}_i| = \sum_{i=1}^{k} r_i(Q_i) = s_1(Q)$$

$\mathcal{J}_2 = \{ Q \subseteq X' \mid Q_i \cap Q_j = \emptyset \ \forall\, 1 \le i < j \le k \}$ ← no $e \in X$ s.t $(e,i) \in Q$ and $(e,j) \in Q$ for some $i \neq j$

$M'' = (X', \mathcal{J}_2)$ is a matroid with rank function $s_2(Q) = \left| \bigcup_{i=1}^{k} Q_i \right|$:

- If $\hat{Q}, Q \in \mathcal{J}_2$ and $\hat{Q} > Q$ then there is some $e \in X$ s.t $(e,i) \in \hat{Q}$ for some i but $(e,j) \notin Q$ for all j.
  Then $Q + (e,i) \in \mathcal{J}_2$ so $\mathcal{J}_2$ satisfies the exchange property
- The maximum size of an independent subset $Q' \subseteq Q$ the number of different elements of X belonging to at least one Q_i so
  $s_2(Q) = \left| \bigcup_{i=1}^{k} Q_i \right|$

Note that we have

$Z \subseteq X$ is partitionable $Z = Z_1 \cup Z_2 \cup \cdots \cup Z_k \quad Z_i \in \mathcal{J}_i$

$\Updownarrow$

$\exists f : Z \to \{1, 2, \ldots, k\}$ s.t $\{(e, f(e)) \mid e \in Z\} \subseteq \mathcal{J}_1 \cap \mathcal{J}_2$

This implies that

$$r(X) = \max\{|Z| \mid Z \text{ is partitionable} \wedge Z \subseteq X\}$$

$$= \max\{|Y| \mid Y \in \mathcal{J}_1 \cap \mathcal{J}_2 \wedge Y \subseteq X'\}$$

$$= \min\{s_1(Q) + s_2(X' - Q) \mid Q \subseteq X'\} \text{ by theorem 13.31}$$

Let $Q \subseteq X'$ be chosen s.t '=' holds in the last line

and take $A = Q_1 \cap Q_2 \cap \cdots \cap Q_k$. Then

$$r(X) = s_1(Q) + s_2(X' - Q) = \sum_{i=1}^{k} r_i(Q_i) + |X - \bigcap_{i=1}^{k} Q_i|$$

$$\geq \sum_{i=1}^{k} r_i(A) + |X - A|$$

This shows that for $A = Q_1 \cap Q_2 \cap \cdots \cap Q_k$ we have

$$r(X) \geq \sum_{i=1}^{k} r_i(A) + |X - A| \quad \text{so}$$

$$r(X) \geq \min_{A \subseteq X} |X - A| + \sum_{i=1}^{k} r_i(A)$$

as claimed $\square$.

It remains to prove that the function $r(X)$ is submodular, that is, $r(X)+r(Y) \geq r(X\cap Y)+r(X\cup Y)$

if this holds then $r: E \to \mathbb{Z}_0$ satisfies the rank axioms

$r(X) \leq |X|$ (taking $A=\emptyset$ shows this)

$Y \subseteq X \Rightarrow r(Y) \leq r(X)$

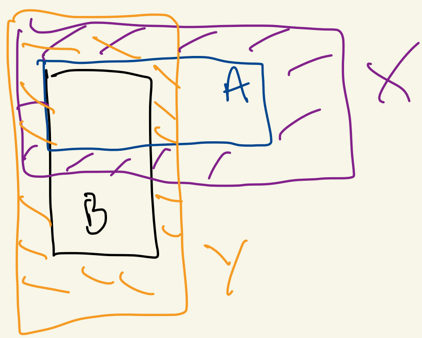
$r(X)+r(Y) \geq r(X\cap Y)+r(X\cup Y) \quad \forall X, Y \subseteq E$

Thus it follows from Theorem 13.10 that the partitionable sets form a matroid since we have

$\mathcal{J} = \{X \mid r(X)=|X|\}$

Given $X, Y \subseteq E$ choose $A \subseteq X$ and $B \subseteq Y$ such that $r(X) = |X-A| + \sum_{i=1}^{k} r_i(A)$ and $r(Y) = |Y-B| + \sum_{i=1}^{k} r_i(B)$

$$
\begin{aligned}
r(X)+r(Y) &= |X-A|+|Y-B| + \sum_{i=1}^{k}(r_i(A)+r_i(B)) \\
&= |(X\cup Y)-(A\cup B)| + |(X\cap Y)-(A\cap B)| + \sum_{i=1}^{k}(r_i(A)+r_i(B)) \\
&\geq |(X\cup Y)-(A\cup B)| + |(X\cap Y)-(A\cap B)| + \sum_{i=1}^{k}(r_i(A\cup B)+r_i(A\cap B)) \\
&\geq r(X\cup Y)+r(X\cap Y)
\end{aligned}
$$



Consequence of proof

- We can find a maximum partitionable subset $Y \subseteq X$ for a given set $X \subseteq E$
- Taking $X = E$ we have an algorithm for finding a maximum partitionable subset of E

Application: Edge-disjoint spanning tree

Given $G = (V, E)$ and a natural number k

Let $M_i = (E, \mathcal{I}_i)$ be the circuit matroid of G $i = 1, 2, \dots k$
(E' is independent if it induces a forest)

Then G has k edge-disjoint spanning trees
$\Updownarrow$ The maximum size of a partitionable subset $X \subseteq E$ is $k(n-1)$

Since $M = (E, \mathcal{I})$ is a matroid when $\mathcal{I} = \{X \subseteq E \mid X \text{ is partitionable}\}$, we can even find a minimum (or maximum) cost collection of $k(n-1)$ edges which partitions into k spanning trees

2nd application : arc-disjoint out-branchings

Given $D=(V,A)$ $s\in V$ and $k\in\mathbb{Z}_+$

Let $M_1 = \bigvee_{i=1}^{k} M_i(D)$ where $M_i(D)=$ circuit matroid of $UG(D)$ (ignore orientation)

and $M_2 = (A, \mathcal{I}_2)$ where

$$A'\in \mathcal{I}_2 \iff d^-_{A'}(v)\le k \;\; \forall v\ne s$$
$$d^-_{A'}(s)=0$$

Then M_1 and M_2 are matroids and we claim that

D has k arc-disjoint out-branchings

$\Updownarrow$

M_1 and M_2 have a common independent set of size $k(n-1)$

$\Downarrow$ If $B^+_{s,1}, \dots, B^+_{s,k}$ are arc-disjoint out-branchings in D then $A(B^+_{s,i})$ is independent in M_i

So $\bigcup_{i=1}^{k} A(B^+_{s,i})$ is independent in M_1

as $d^-_{B^+_{s,i}}(x)=1 \quad \forall x \neq s$ and $d^-_{B^+_{s,i}}(s)=0$

we also have that $\bigcup_{i=1}^{k} A(B^+_{s,i})$ is independent in M_2

To see $\Uparrow$ we observe if $|A'| = k(n-1)$ and

1. A' induces k edge-disjoint spanning trees in $UG(D)$

+ 2. $d^-_{A'}(x)=k \quad \forall x \neq s$

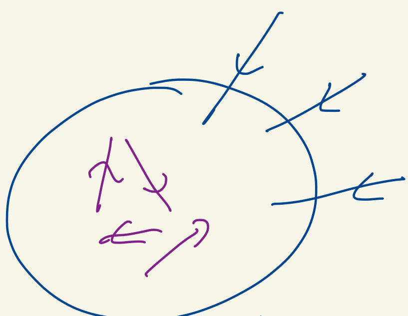
Then $d^-(U) \geq k$ for all $U \subseteq V-s$, because

$$k|U| = \sum_{x \in U} d^-_{A'}(x) \leq k(|U|-1) + d^-(U)$$

$\Downarrow$

$$d^-(U) \geq k$$

as A' induces k forests inside U there are at most $(|U|-1)k$ A' arcs inside U



U

Thm (Edmonds matroid partition theorem)

Let $M_i = (E, \mathcal{I}_i)$ $i = 1, 2, \dots, k$ be matroids over E and let X be a maximum size partitionable set, then

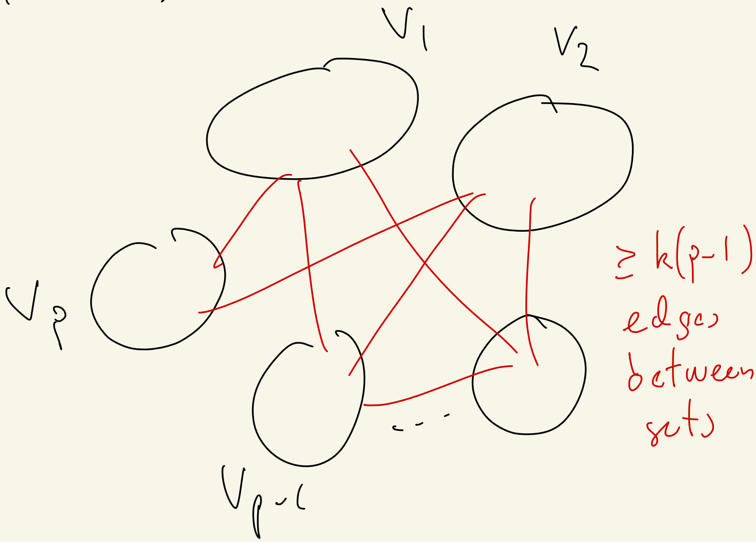
$$|X| = \min_{A \subseteq E} |E - A| + \sum_{i=1}^{k} r_i(A)$$

Corollary (Tutte)

A graph G has k edge-disjoint spanning trees if and only if

$$\varepsilon(\mathcal{F}) \geq k(|\mathcal{F}| - 1)$$ for all partitions $\mathcal{F}$ of V

Necessity is clear



<u>Sufficiency</u> assume (I) holds $\forall$ partitions

We claim $\displaystyle\min_{A \subseteq E} |E-A| + \sum_{i=1}^{k} r(A) \geq k(n-1)$

where r is the rank function of the cycle matroid of G

<u>note</u> $r(E') = \max$ # edges in a forest in E'

$= n -$ # components in $G[E']$

Let $A \subseteq E'$ be chosen such that

$$R = |E - A| + \sum_{i=}^{k} r(A) \quad \text{is minimum}$$

then $R = |E - A| + k(n - \#\text{component in } G[A])$

by ($\square$)

$$\geq \underbrace{k(\#\text{components in } G[A]) - 1)}_{} + k(n - \#\text{comp in } G[A])$$

$$= k(n-1)$$

So by Edmonds theorem the rank of

$$\bigvee_{i=1}^{k} M(G) \text{ is } k(n-1)$$

implying that G has k edge-disj sp trees.